THEORETICAL & APPLIED MECHANICS LETTERS 3, 063011 (2013)

Stability analysis of the active control system with time delay using IHB method

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(Received 1 October 2013; accepted 3 October 2013; published online 10 November 2013)

Abstract In this paper, the incremental harmonic balance method is employed to solve the periodic solution that a vibration active control system with double time delays generates, and the stability analysis of which is achieved by the Poincare theorem. The system stability regions can be obtained in view of time delay and feedback gain, the variation of which is also studied. It turns out that along with the increase of time delay, the active control system is not always from stable to unstable, and the system can be from stable to unstable state, whereas the system can be from unstable to stable state. The extent that the two times delays impact on the system stability region is mainly related to the relative magnitude of the two feedback gains. The system can maintain the stable state under the condition of the well-matched feedback gains. The results can provide evidence to design the control strategy of time-delayed feedback. [doi:10.1063/2.1306311]

Keywords active control system, double time delays, IHB method, stability

Time delay broadly occurs in the active control systems[1,2] and it brings some difficulties with the structural vibration control systems. Some study results indicate that time delay can lead to the unstable of a vibration active control system. So researching the stability of active vibration control systems with time delay is very necessary.[3,4]

With regards to the stability study of the system with time delay, two main analytical methods are put forward currently. One is the time domain method and the other is the frequency domain method. The time domain method is performed by constructing an appropriate Lyapunov function. Wu[5] obtains a delay-dependent stability of the open-loop fuzzy system using a new fuzzy Lyapunov–Krasovskii functional and also studies the stability of the discrete-time Takagi and Sugeno fuzzy systems with the state time delay. Liu and Wang[6] establish several exponential stability criteria by employing the theorem of Lyapunov functional to study the exponential stability of the time-delay impulsive systems. Chen[7] presents a Takagi–Sugeno fuzzy model for the modeling and stability analysis of oceanic structures using the Lyapunov method. These researches are all based on the Lyapunov stability theory. But in practical problems, it is difficult to establish Lyapunov function; therefore, the use of the time domain method is very restricted.

In addition to the method mentioned above, another one is called frequency domain method in which the eigenvalue method is the most widely used for the analysis of the time-delay systems. Wang and Hu[8] calculate the rightmost root of the eigenfunction, establishing a new stabilization criterion and determining the admissible values of the feedback gains and delays with the given effective procedure.[9] Yi et al.[10] raise a question of feedback controller by means of the eigenvalue assignment for linear time-invariant system of linear delay differential equations with a single delay. Bai et al.[11] use a multi-step hybrid method to solve the multi-input partial quadratic eigenvalues assignment problem with both the system matrices and the receptance measurements. The methods adopted by those references require solving every eigenvalues of the eigenfunction, however, the practical eigenequation of the controlled system with time delay is a transcendental equation, and thus how to figure out its accurate eigenvalues of the transcendental equation is the core issue which must be settled successfully.

In this paper, the incremental harmonic balance (IHB) method is used to solve the periodic solution that a vibration active control system with double time delays generates, then the stability analysis of which is achieved by the Poincare theorem. With respect to different matches of time delay and feedback gain, the system stable regions are found out.

The strongly nonlinear dynamics control system with the displacement feedback and the velocity feedback is taken into account in this paper. The external excitation is assumed as harmonic excitation. Then the motion equation is given by

$$\begin{aligned}\ddot{y}(\tau) + 2\xi\dot{y}(\tau) + gy(\tau) + ky^3(\tau) &= \\ F(\tau) + s_1 y(\tau - u_1) + s_2\dot{y}(\tau - u_2),\end{aligned} \tag{1}$$

where ξ is the damping coefficient, g is the restoring force coefficient, s_1 and u_1 are the displacement feedback gain and time delay, s_2 and u_2 are the velocity feedback gain and time delay, $F(\tau)$ is the external excitation and taken as $F(\tau) = f\cos(\omega\tau)$, ω is the exciting frequency, and f is the excitation amplitude. The time

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delay differential equation above is solved by the IHB method in this paper.

To nondimensionalize Eq. (1), we let

$$t = \omega\tau,\ y(\tau) = x(t),\ F(\tau) = f \sin t. \tag{2}$$

Substituting Eq. (2) into Eq. (1), we can obtain

$$\omega^2\ddot{x} + 2\xi\omega\dot{x} + gx + kx^3 = f\sin t + s_1 x(t - \omega u_1) + \omega s_2 \dot{x}(t - \omega u_2). \tag{3}$$

Let x_0 and ω_0 represent a certain state in vibration, then the adjacent state can be represented as incremental forms

$$x = x_0 + \Delta x, \quad \omega = \omega_0 + \Delta\omega. \tag{4}$$

Equation (4) is substituted in Eq. (3), and the incremental Eq. (5) is obtained after ignoring the high order quantity. It should be noted that Δx, $\Delta\omega$ are the unknown quantities.

$$\begin{aligned}&\omega_0^2\Delta\ddot{x} + 2\xi\omega_0\Delta\dot{x} + (g + 3kx_0^2)\Delta x = R - \\ &\quad 2\omega_0\Delta\omega\ddot{x}_0 - 2\xi\Delta\omega\dot{x}_0 + \\ &\quad s_1[x_0(t - \omega_0 u_1 - \Delta\omega u_1) + \\ &\quad \Delta x(t - \omega_0 u_1 - \Delta\omega u_1)] + \\ &\quad s_2(\Delta\omega + \omega_0)\dot{x}_0(t - \omega_0 u_2 - \Delta\omega u_2) + \\ &\quad s_2\omega_0\Delta\dot{x}(t - \omega_0 u_2 - \Delta\omega u_2) - \\ &\quad s_1 x_0(t - \omega_0 u_1) - s_2\omega_0\dot{x}_0(t - \omega_0 u_2)\end{aligned} \tag{5}$$

and

$$\begin{aligned}R = {}& f\sin t + s_1 x_0(t - \omega_0 u_1) + s_2\omega_0\dot{x}_0(t - \omega_0 u_2) - \\ & (\omega_0^2\ddot{x}_0 + 2\xi\omega_0\dot{x}_0 + gx_0 + kx_0^3),\end{aligned} \tag{6}$$

where R is imbalance force, and theoretically when $R = 0$, x_0 and ω_0 will be the exact solutions.

Supposing the solution of the Eq. (5) is

$$\begin{aligned}x_0 &= \sum_{n=0}^{\infty}[a_n\cos(nt) + b_n\sin(nt)], \\ \Delta x &= \sum_{n=0}^{\infty}[\Delta a_n\cos(nt) + \Delta b_n\sin(nt)],\end{aligned} \tag{7}$$

Eq. (7) is substituted in Eq. (5). Let the harmonic term coefficients be equal value. Then the equation is obtained as

$$\boldsymbol{K}\Delta\boldsymbol{a} = \boldsymbol{R}_1 + \boldsymbol{R}_2\Delta\omega, \tag{8}$$

where $\boldsymbol{K}$ is $n \times n$ dimension matrix, $\Delta\boldsymbol{a}$, $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ are $n \times 1$ column vectors.

Let $\Delta\omega = 0$, give initial values to the harmonic term coefficients, and substitute the initial values in Eq. (8) to get Δa. Then a is replaced by $a + \Delta a$ which is substituted in Eq. (8) to get the new value of Δa. Based on the substitution, the equation is iterated circularly until R is up to the appointed precision.

The system periodic solution solved by the IHB method can not determine the dynamic system stably or unstably. For this reason, the Poincare theorem is employed herein to judge the stability of the solution and it is expressed as follows.

Supposing C_0 is the periodic solution of the autonomous system which can be written as

$$\begin{aligned}\dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y),\end{aligned} \tag{9}$$

where $C_0 : x = \varphi_0(t),\ y = \psi_0(t)$.

The period of $\varphi_0(t)$ and $\psi_0(t)$ is T. Then the integral expression is

$$h_c = \frac{1}{T}\int_0^T \left[\frac{\partial P}{\partial x}(\varphi_0, \psi_0) + \frac{\partial Q}{\partial y}(\varphi_0, \psi_0)\right]\mathrm{d}t, \tag{10}$$

and it is called the characteristic index of C_0.

According to the Poincare theorem, the periodic solution is stable or unstable when the characteristic index is negative or positive.

Theoretically speaking, the Poincare theorem is applied to the autonomous system. Hence, how to improve the method so as to hold for the non-autonomy system is the key problem which should be considered and settled effectively.

Considering the forced vibration of the nonlinear system, the motion equation is shown as

$$\ddot{x} + f(x, \dot{x}) = F(t). \tag{11}$$

Supposing x_0 is the solution, Δx is the small disturbance quantity given by

$$x = x_0 + \Delta x. \tag{12}$$

Equation (12) is substituted in Eq. (11) where the high-order quantity is omitted, so the incremental equation (13) is obtained as

$$\Delta\ddot{x} + f(\Delta x, \Delta\dot{x}) = 0. \tag{13}$$

Equation (13) is a perturbation equation of Eq. (11). Based on the motion stability theory, the stability of the periodic solution of the Eq. (11) corresponds to the stability of the periodic solution of the Eq. (13).[12]

The next is to deal with the time delay term of the Eq. (5), the solution of which can be obtained by using the IHB method expressed as

$$x = \sum_{n=0}^{\infty}[a_n\cos(n\omega t) + b_n\sin(n\omega t)]. \tag{14}$$

First and second derivations of Eq. (14) induce the expressions $x'(\mathrm{d}x/\mathrm{d}t)$ and $x''(\mathrm{d}^2x/\mathrm{d}t^2)$. Likewise, the expressions $x(t-u), x'(t-u), ..., x^{(n)}(t-u)$ are induced in turn as

$$\begin{aligned}&x(t-u) = m_{10}x(t) + m_{11}x'(t) + \\ &\quad m_{12}x''(t) + ... + m_{1n}x^{(n)}(t), \\ &x'(t-u) = m_{20}x(t) + m_{21}x'(t) + \\ &\quad m_{22}x''(t) + ... + m_{2n}x^{(n)}(t), \\ &x''(t-u) = m_{30}x(t) + m_{31}x'(t) + \\ &\quad m_{32}x''(t) + ... + m_{3n}x^{(n)}(t), \\ &\cdots \\ &x^{(n)}(t-u) = m_{n0}x(t) + m_{n1}x'(t) + \\ &\quad m_{n2}x''(t) + ... + m_{nn}x^{(n)}(t).\end{aligned} \tag{15}$$

Equation (14) is substituted in Eq. (15). The undetermined coefficients $m_{10}, m_{11}, ..., m_{1n}, ..., m_{nn}$ can be obtained according to the harmonic balance theory. Then the characteristic index of the periodic solution of Eq. (5) can be got by Eq. (10). Accordingly the stability of the periodic solution can be determined by using the Poincare theorem.

Letting $\omega = f = 1$, $\xi = 0.02$, $s_1 = 0.4$, $s_2 = 0.128\,1$ in Eq. (3), two time delays are taken as the control parameters and then the stability analysis of the periodic solution is carried out using the Poincare theorem. To certify the validity of the proposed approach, the results are compared with the numerical solution which is obtained by adopting the adaptive variable step size Runge–Kutta–Felberg method. The phase planes are shown in Figs. 1 and 2.

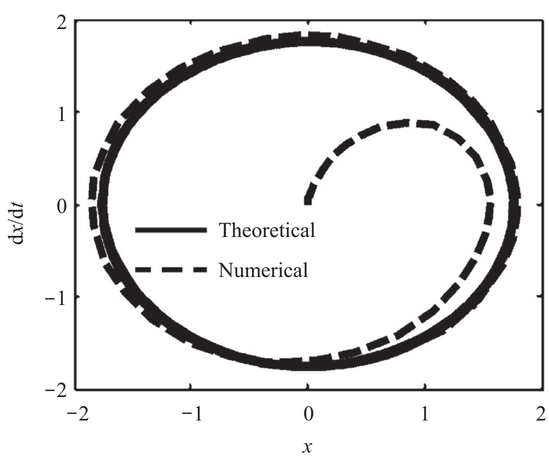

Fig. 1. Phase plane with $u_1 = 1.5$, $u_2 = 3$.

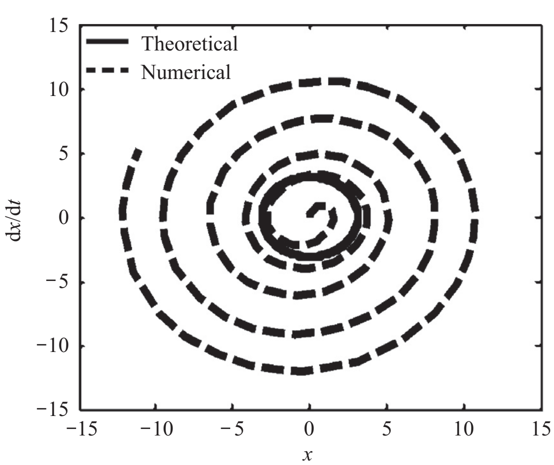

Fig. 2. Phase plane with $u_1 = 10$, $u_2 = 3$.

Figure 1 illustrates the phase plane with $u_1 = 1.5$, $u_2 = 3$. Based on the theory discussed previously, we can get the characteristic index of the periodic solution $C_0 = -0.540\,7$. The periodic solution is stable according to the Poincare theorem which is also proofed (Fig. 1). Likewise when $u_1 = 10$, $u_2 = 3$, the characteristic index of the periodic solution is $C_0 = 0.074\,4$, which accounts for the conclusion that the system is unstable, which is also verified by the numerical result as shown in Fig. 2.

Similarly, letting $\omega = f = 1$, $\xi = 0.02$ in Eq. (3), two feedback gains are taken as the control parameters and then the stability analysis of the periodic solution is carried out by using Poincare theorem. After collecting the results of different time-delay matches, the stability regions corresponding to the characteristic indexes are drawn consequently in Fig. 3.

As shown in Fig. 3(a), the variational stability areas display the zonal distribution with the displacement feedback delay, with $s_1 = 0.4$, $s_2 = 0.128\,1$. The unstable and stable regions appear alternately. The influence of the velocity feedback delay impacting on the system stability appears only in the vicinity of the boundary between the unstable region and stable region, which barely has effect on the system stability. On the contrary, the influence of the displacement feedback delay on the system stability region is obviously greater than the velocity feedback delay and plays the decisive role.

From Figs. 3(b) to 3(e), it shows that the effect of the velocity feedback delay on the system stability is increasing gradually in the condition that the relative magnitude of the two feedback gains varies. The displacement feedback gain is invariant and the velocity feedback gain is increasing from less to more than the displacement feedback gain. We can see from the Fig. 3(e) that the velocity feedback delay on the system stability has played the decisive role in converse to the Fig. 3(a). However, the system stability is determined by whether the crossing point of the two feedback delays falls into the region S, that is to say, either of the two delays can not entirely determine the system stability. As a consequence, the system will be stable only if the two delays have a good combination.

According to the phenomena derived from the above results, some laws can be attained. The distribution of the system stability region is mainly related to the relative magnitude of the two feedback gains. Meanwhile the system stability can be mainly related to the relative magnitude of the two time delays under the condition of the two specified feedback gains.

The stability of the active vibration control system with double time delay is analyzed in this paper. The results show that along with the increase of time delay, the active control system is not always from stable to unstable, and the system can be from stable to unstable state, whereas the system can be from unstable to stable state. The extent of the two times delays impacting on the system stability region is mainly related to the relative magnitude of the two feedback gains. The stability regions are controlled by the relatively important feedback delay. These laws can provide evidence to design an effective control strategy of a time-delay system.

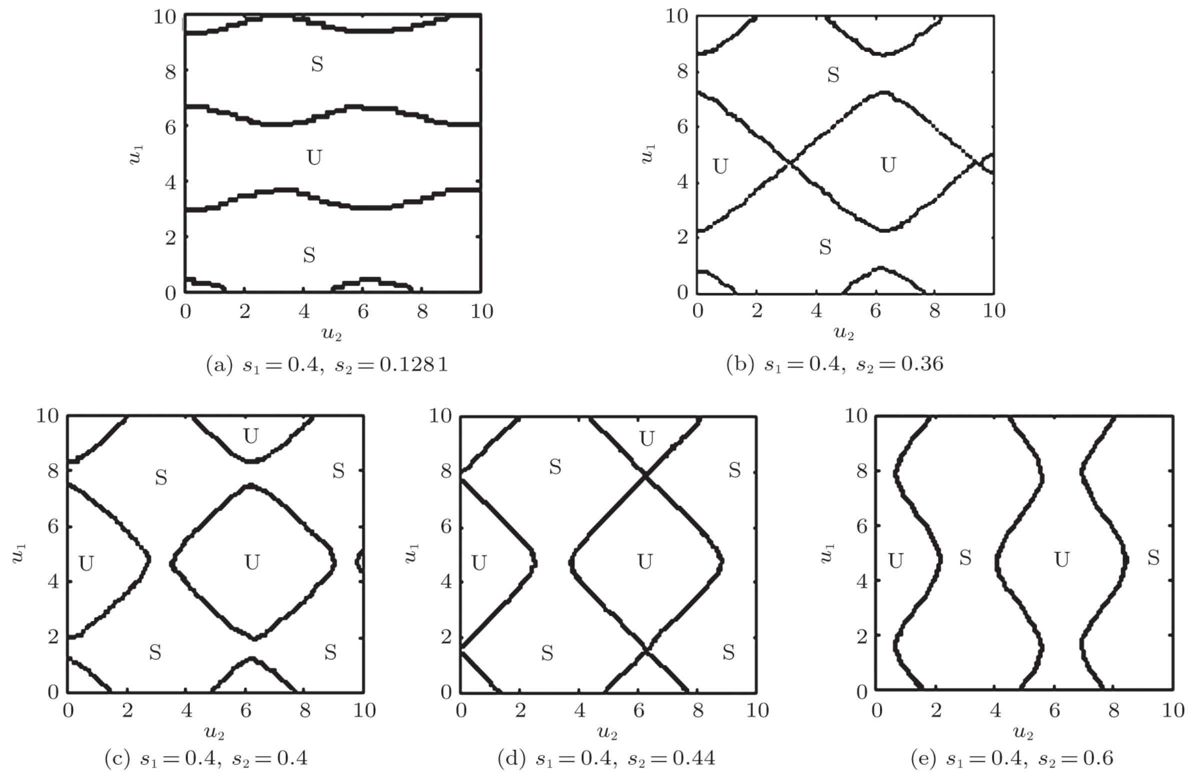

Fig. 3. System stability region. "U" represents the unstable region; "S" represents the stable region.

This work was supported by the National Natural Science Foundation of China (11172226).

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